THE JOURNAL OF VISUALIZATION AND COMPUTER ANIMATION
J. Visual. Comput. Animat. **9**, 95–107 (1998)

Periodic Motion Synthesis and Fourier Compression

MIKIO SHINYA,[1]* TAKEAKI MORI[2] AND NORIYOSHI OSUMI[3]
[1]*NTT Human Interface Laboratories, 1-1 Hikari-no-oka, Yokosuka-shi, Kanagawa-ken 239, Japan*
[2]*NTT America, Inc., 700 E El Camino Real #170, Mountain View, CA 94040, U.S.A.*
[3]*NTT System Services Department, 700 E El Camino Real #170, Mountain View, CA 94040, U.S.A.*

SUMMARY

Periodic motion is an important class of motion to synthesize, but it is not easy to compute it robustly and efficiently. In this paper we propose a simple, robust and efficient method to compute periodic motion from linear equation systems. The method first calculates the response of the system when an external periodic force is applied during one period, and then sums up the periodically shifted versions of the system response to provide the periodic solution. It is also shown that Fourier decomposition is very effective to compress the motion data without a drop in visual fidelity. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: physically based modelling; linear systems; stationary motion; wind models

1. INTRODUCTION

Synthesizing realistic motion is one of the most challenging goals in computer animation. Many studies have been made towards this goal, and the physically based approach has been widely recognized to be a promising way to create realistic motion (Reference 1 is a good introduction). In the physically based approach the motion synthesis process is formulated as a dynamic simulation process. Such a process is usually represented by a linear equation system, which can be solved, for example, by Euler's method. Physically based simulation is a powerful method, but solving a system of equations requires a lot of computation time, especially for complex systems, and it is difficult to achieve real-time visual simulations such as virtual reality applications.

If the motion is stationary and periodic, we can precompute the motion patterns and make an animation by repeatedly applying the precomputed pattern in image generation, as in the case of repeatedly applying periodic texture patterns in rendering. Some stochastic motion in nature can be modelled using its intrinsic periodicity,

* Correspondence to: Mikio Shinya, NTT Human Interface Laboratories, 1-1 Hikari-no-oka, Yokosuka-shi, Kanagawa-ken 239, Japan

CCC 1049–8907/98/020095–13$17.50

Received 1 September 1996
Revised 25 September 1997

such as water flow in water streams, waves at the seashore, swaying trees in wind, and so on. However, most physically based methods are performed in the time domain, typically by Euler's method, which does not always provide a stationary periodic solution owing to errors. Another problem with this mapping approach is that motion patterns require a huge memory for complex dynamic systems if we attempt to save all motion data.

This paper describes an efficient and stable algorithm to compute the periodic responses of linear systems when periodic external forces are applied. First, the algorithm calculates the response of the system when the force is applied only during one particular period (e.g. $0 < t < T$). The response $p(t)$ is computed until the motion converges to zero. Second, the algorithm calculates the summations of shifted patterns of the computed response, $\Sigma p(t - nT)$, for $n = 0, 1, 2, \ldots$. This summation is, in turn, the periodic solution of the original problem. The validity of this approach can be easily proved from the superposition feature of linear systems. Further, we apply Fourier decomposition to the simulation results to compress the data amount without loss of visual fidelity. The method is applied to a physically based stochastic motion synthesis of trees. The experimental results are very encouraging and confirm that just a small number of Fourier components can reconstruct visually realistic motion.

2. LINEAR SYSTEMS AND PERIODIC MOTION

2.1. Basics

Most applications of physically based animation are formulated as linear equation systems. For example, damped spring–mass systems can be modelled by

$$\left(M\frac{\mathrm{d}^2}{\mathrm{d}t^2} + C\frac{\mathrm{d}}{\mathrm{d}t} + K\right)\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t) \end{pmatrix} \tag{1}$$

where M, C and K are $n \times n$ matrices known as the inertia matrix, the damping matrix and the stiffness matrix, respectively. If the applied force $f_i(t)$ is a periodic function with period T, i.e.

$$f_i(t + T) = f_i(t) \tag{2}$$

then the stationary response of the system is periodic as well.

For simplicity, let us first consider the one-dimensional system described by

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k\right)x(t) = f(t) \tag{3}$$

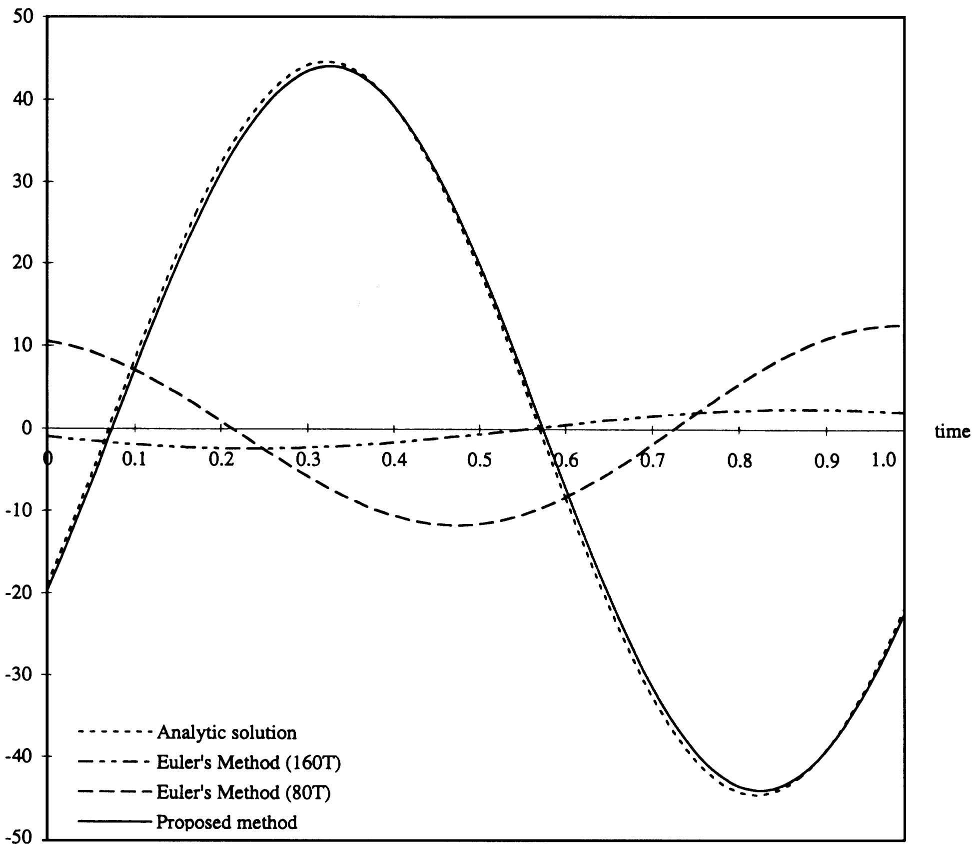

Figure 1. Solutions for a single spring–mass system

where m is the mass, c is the damping constant and k is the stiffness constant. There are three typical methods to solve the problem.

Direct method

The simplest way to get the solution is to apply Euler's method, which approximates differentials by displacements over a finite time step Δ_t:

$$v(t + \Delta_t) = \frac{(f(t) - cv(t) - kx(t))\Delta_t}{m} + v(t) \tag{4}$$

$$x(t + \Delta_t) = v(t)\Delta_t + x(t) \tag{5}$$

At the beginning the solution depends strongly on the initial condition and is not periodic. In theory, after processing a sufficient number of time steps, the solution tends to the stationary periodic solution. In practice, however, discretization errors may cause unstable behaviour, and convergence cannot be guaranteed in critical cases.

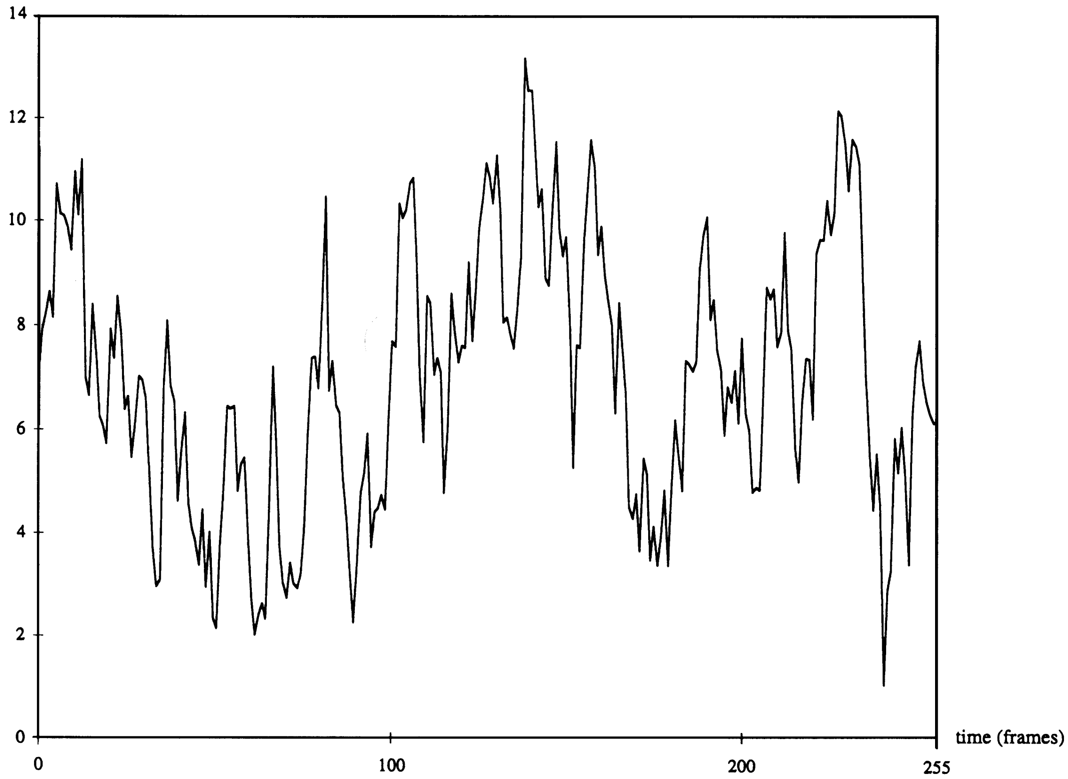

Figure 2. Wind velocity

Fourier method

Another possible approach is to apply the Fourier method, which is also known as the frequency response method in linear circuit theory. Applying Fourier transfer to equation (3), we have

$$A(\omega)\hat{x}(\omega) = \hat{f}(\omega) \tag{6}$$

$$\hat{x}(\omega) = A^{-1}\hat{f}(\omega) \tag{7}$$

where $\hat{x}$ denotes the Fourier transform of the function $x(t)$, ω is the angular frequency and $A(\omega)$ is the frequency response of the system. For the single damped oscillator example, the response can be calculated as

$$A(\omega) = m\omega^2 - \mathrm{i}c\omega + k$$

Thus, applying the inverse Fourier transform

$$x(t) = \int (A^{-1}\hat{f})\exp(-\mathrm{i}\omega t)\mathrm{d}\omega$$

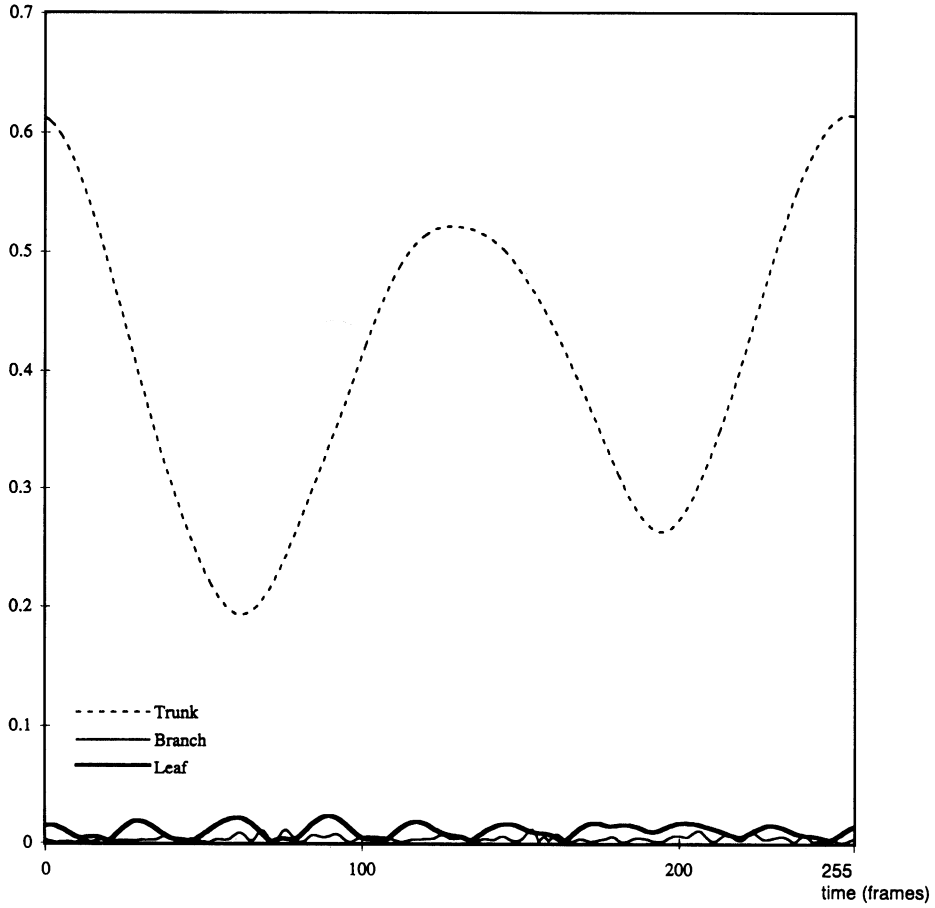

Figure 3. Responses of a trunk, a branch and a leaf

yields the stationary solution. This can be computed by FFT, and the solution is guaranteed to be stable and periodic.

This method works fine in one-dimensional cases because A^{-1} is a simple scalar division. In multi-spring cases, however, the response A is a matrix, so calculation of A^{-1} requires matrix inversion, which is generally hard to compute, especially for large matrices.

Impulse response method

An equivalent computation can also be performed in the time domain, known as the impulse response method. The impulse response $h(t)$ is the response of the system when a unit impulse force $\delta(t)$ is applied to the system. For the system described by equation (3), $h(t)$ satisfies

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k\right)h(t) = \delta(t) \tag{8}$$

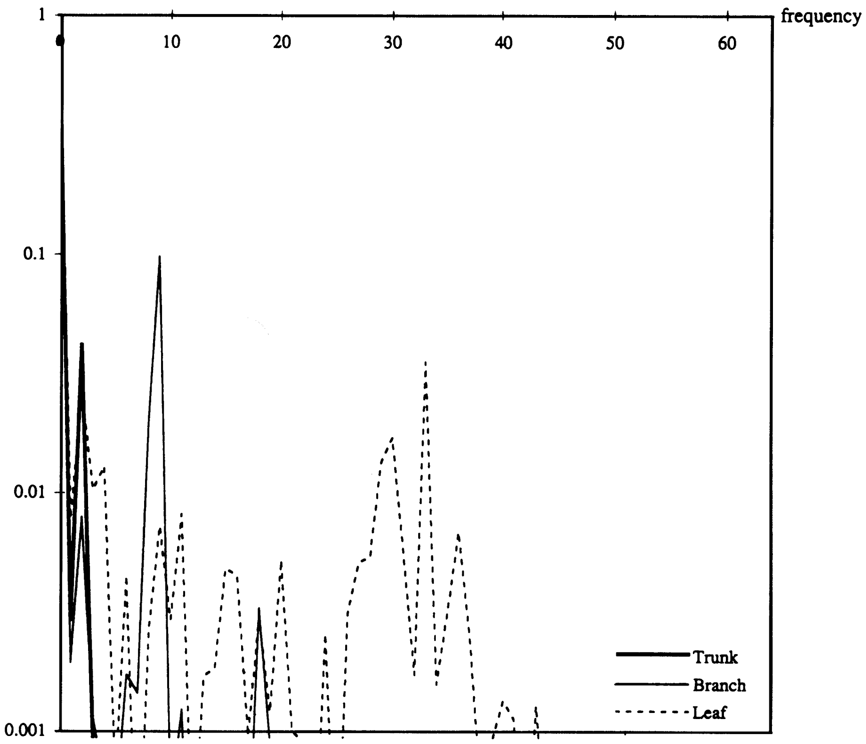

Figure 4. Power spectra of responses

$$h(0) = 0 \tag{9}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}(0) = 0 \tag{10}$$

From the definition of the delta function we can represent the external force $f(t)$ as a convolution with the delta function:

$$f(t) = \int f(t')\delta(t - t')\mathrm{d}t' \tag{11}$$

From equations (8) and (11) we have

$$f(t) = \int f(t')\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k\right)h(t - t')\mathrm{d}t' \tag{12}$$

$$= \left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k\right)\left(\int f(t')h(t - t')\mathrm{d}t'\right) \tag{13}$$

This means that the integral

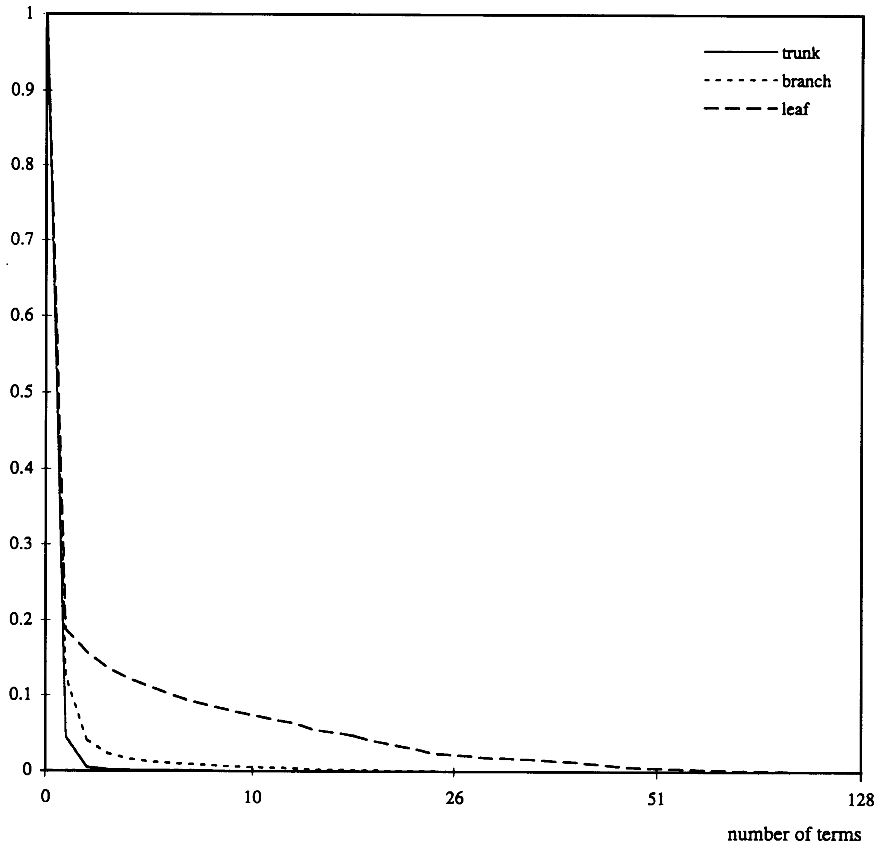

Figure 5. Number of Fourier components and error

$$x(t) = \int_0^\infty f(t - t')h(t')\mathrm{d}t' \tag{14}$$

is the solution of equation (3). The integral in equation (14) is an infinite integral, but it is generally true that the impulse response functions of stable systems rapidly tend to zero with t. Therefore, if the response after $t > t_c$ can be neglected, i.e.

$$h(t) \approx 0 \quad \text{for } t > t_c$$

the integral can be reasonably approximated by a finite integral as

$$x(t) \approx \int_0^{t_c} f(t - t')h(t')\mathrm{d}t' \tag{15}$$

The result is assured to be stable and periodic as in the case of the Fourier method.

It is known that the impulse response $h(t)$ is equal to the inverse Fourier transform

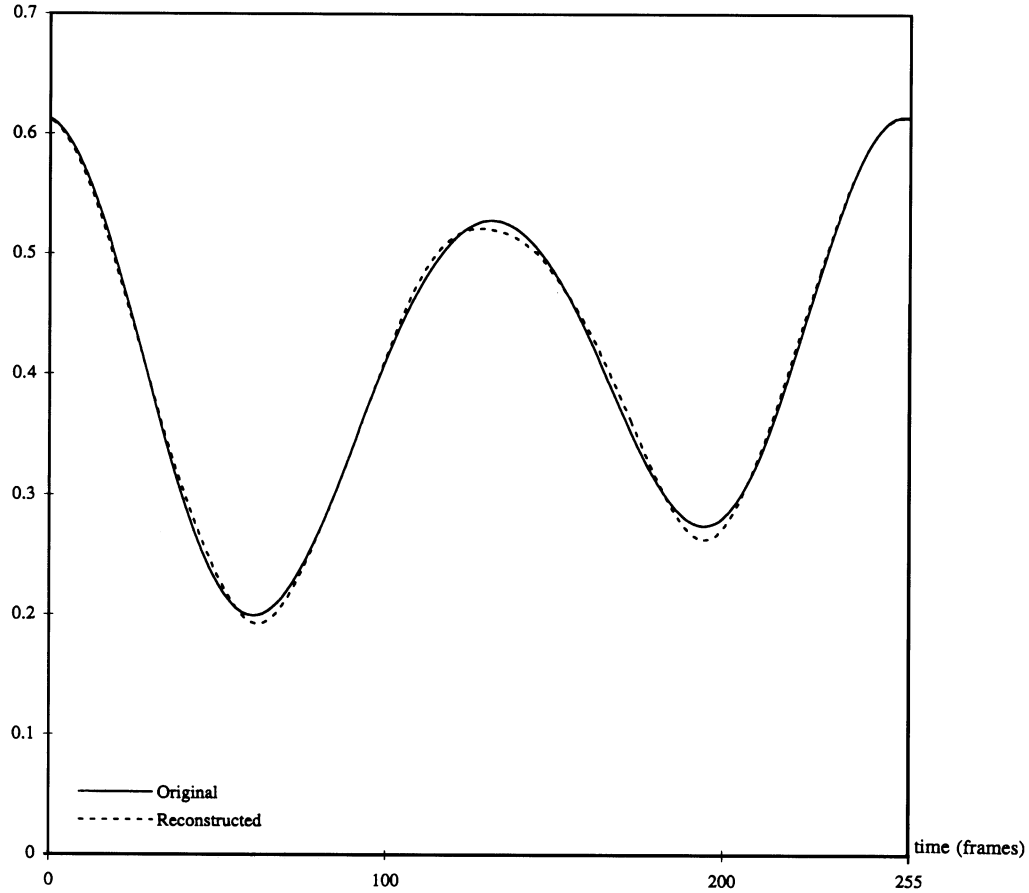

Figure 6. (a) Reconstructed response (trunk). (b) Reconstructed response (branch). (c) Reconstructed response (leaf)

of the frequency response $H(\omega)$, so it can be calculated from H via FFT. Of course, it is also possible to solve equation (8) directly by Euler's method.

There is also another problem with large systems. In the case of n degrees of freedom (DOFs) the impulse response is an $n \times n$ matrix and n^2 functions have to be computed, which can be very expensive for high-DOF systems.

2.2. New method for periodic motion

Our idea is to combine the direct method and the impulse response method so that the stationary solution can be efficiently and robustly obtained for high-DOF systems. We first consider applying the external force $f(t)$ for only one period, e.g. $0 < t < T$, and then let the system move by itself. We denote this truncated force by

$$f_T(t) = \begin{cases} f(t) & \text{if } 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

For the single spring–mass system the response $p(t)$ satisfies

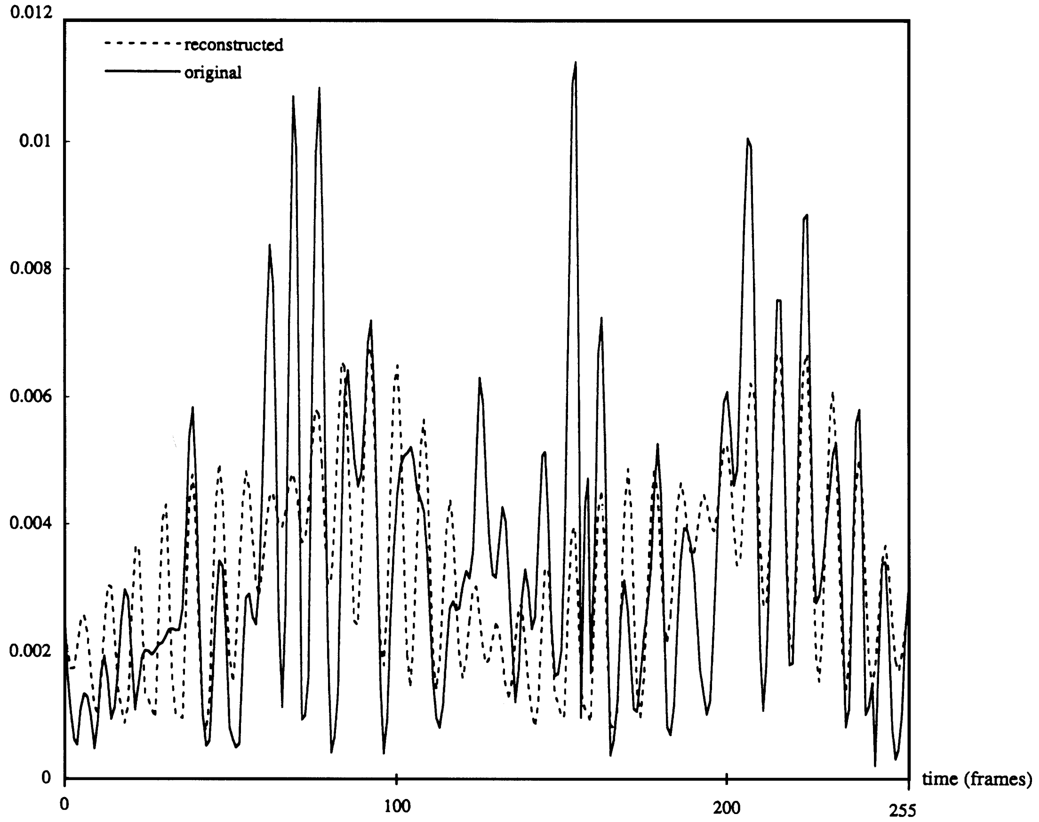

Figure 6. (b) Continued

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k\right)p(t) = f_T(t) \tag{16}$$

$$p(0) = 0 \tag{17}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t}(0) = 0 \tag{18}$$

Since $f(t)$ is a periodic function with period T, $f(t)$ can be represented by an infinite sum of f_T:

$$f(t) = \sum_{n=-\infty}^{\infty} f_T(t - nT) \tag{19}$$

Therefore, if we set

$$x(t) = \sum_{n=-\infty}^{\infty} p(t - nT) \tag{20}$$

the summation $x(t)$ satisfies the original equation:

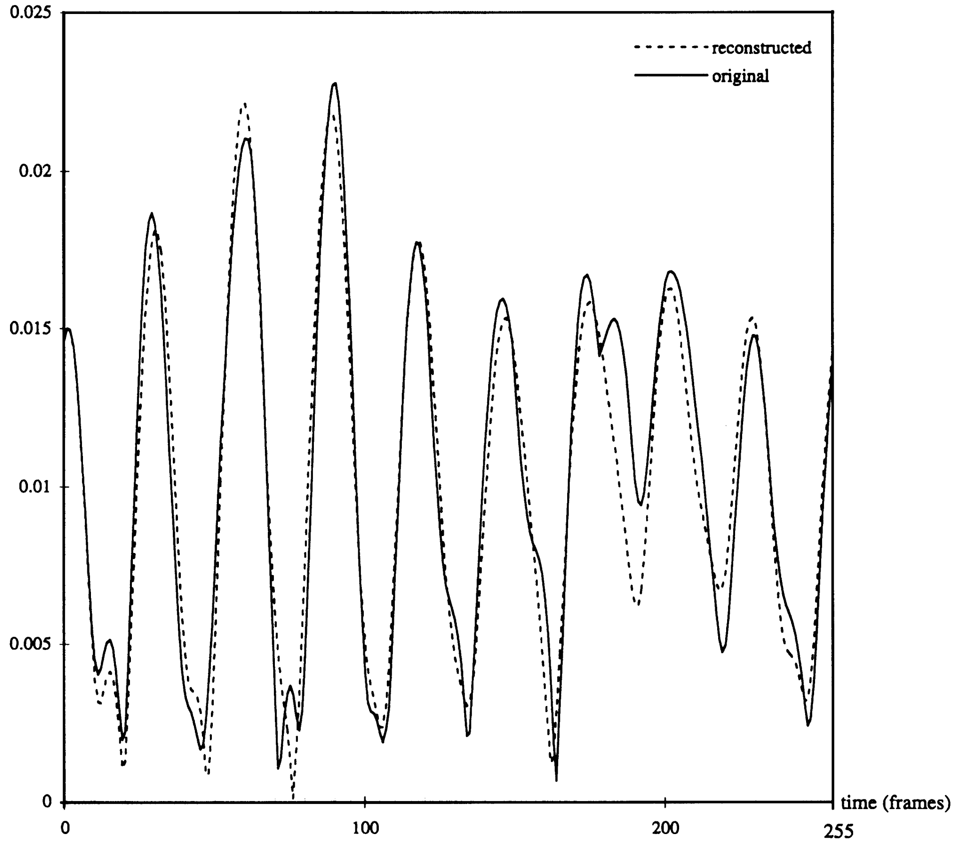

Figure 6. (c) Continued

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2}+c\frac{\mathrm{d}}{\mathrm{d}t}+k\right)x(t)=\sum_{n=-\infty}^{\infty}f_T(t-nT)=f(t)$$

Thus we can get the solution $x(t)$ of equation (3) by equation (20), which is the superposition of shifted versions of the response $p(t)$. Equation (20) involves an infinite sum, but, like the impulse response $h(t)$, the response $p(t)$ rapidly tends to zero and the sum is well approximated by a finite summation. The response $p(t)$ can be easily calculated by a direct method such as Euler's method. The solution resulting from the superposition is stable and guaranteed to be periodic.

As is easily proved, this holds for any linear system. The advantage of this method over the Fourier method and the impulse response method is its efficient computation for complex systems. Unlike the impulse response method, we only have to compute n functions for n-DOF systems.

The procedure to compute $x(t)$ is straightforward, as outlined below.

1. Compute $p(t)$ by Euler's method.
2. Find an integer number m such that $|p(t)|<\theta$ for $t>mT$ and for a given threshold θ.
3. Calculate the sum $x(t)=\Sigma_0^m p(t-mT)$ for $0<t<T$.

3. EXPERIMENT

3.1. Single mass–spring system

First we applied the method to a single spring–mass system, described by equation (3), and made comparison with the analytic solution. The applied force is a unit sinusoidal function

$$f(t) = \sin(2\pi t/T) = \sin(\omega t)$$

In this case we have the analytic solution $x_a(t)$:

$$x_a(t) = A\sin(\omega t + \phi) \tag{21}$$

$$A = [(k - m\omega^2)^2 + (\omega c)^2]^{-\frac{1}{2}} \tag{22}$$

$$\phi = -\tan^{-1}[c\omega/(k - m\omega^2)] \tag{23}$$

When the frequency ω is close to the resonance frequency of the system ($\omega_0 = \sqrt{k/m}$), a direct application of Euler's method to equation (3) with the periodic external force can result in large unstable errors, even with small time steps. Figure 1 shows the solutions obtained by Euler's method and the proposed method. The analytic solution is also plotted for reference. The parameters used are $\omega = 1{\cdot}0$ ($T = 2\pi$), $m = 1$, $\omega_0 = 1{\cdot}01$ ($k = 1{\cdot}01$), $c = 0{\cdot}01\ \omega_0^2$ and time step $\Delta_t = T/10{,}000$. The result from Euler's method shows a large difference in the computed responses for the 80th period ($80T < t < 81T$) and for the 160th ($160T < t < 161T$) period, demonstrating the unsuitability of directly applying Euler's method. Also, they both involve large errors. The L^2-error ratio defined by

$$\sum_{j=0}^{T/\Delta_t} (x_a(j\Delta_t) - x(j\Delta_t))^2 / \sum_j (x_a(j\Delta_t))^2$$

was 5·07 for the 80th period and 1·40 for the 160th period. On the other hand, the result from the proposed method is stable and agrees with the analytic solution to a large extent. The measured error was only 0·0008.

3.2. Bamboo swaying

Next we applied the method to a more complicated system. Motion under the influence of wind has attracted many researchers[2–6] and it is known that stationary turbulence can be represented by a periodic model.[7] This motivated us to apply our method to the synthesis of tree-swaying motion.[4] A sample image is shown in Plate 1. In this application the external force comes from a stochastic wind field, which is modelled as a power spectrum. The wind field is obtained by applying FFT to the spectrum, so the force is periodic with period T. A sample of the wind velocity is shown in Figure 2.

By applying a modal analysis technique, the system can be formalized as a second-order differential equation system (for details see Reference 4), which was then solved by the proposed method. This example is a stand of bamboo consisting of approximately 3000 branches (9000 DOFs). The period T was set to 256 frames (8·53 s). Some sample results for a trunk, a branch and a leaf are shown in Figure 3.

3.3. Fourier decomposition

Our method is efficient in computing physically based periodic motion, but the computed response function requires a large storage space. For example, when there are 10,000 DOFs and the period is 1000, 10^7 data must be stored for just one tree. If there are 100 trees in a scene, more than 1 GB of memory is required even when 1 byte is allocated to represent a datum. However, if the power spectrum of the motion has a narrow frequency band, we can efficiently compress the data by Fourier decomposition. Figure 4 shows the power spectra of the responses presented in Figure 3. As seen in the figure, most energy is distributed in the low-frequency domain. Thus the n-best components can be a good approximation of the original response. Figure 5 shows the L^2-error defined by

$$e = \Sigma|x(t) - x_f(t)|^2/\Sigma|x(t)|^2$$

where $x_f(t)$ is the Fourier composition of $x(t)$ from the n-largest frequency components $\{\omega_i\}_{i=1,n}$, represented by

$$x_f(t) = \sum_{j=1}^{n} \hat{x}(\omega_j)\exp(-\omega_j t)$$

The figure suggests that we can accurately reconstruct the periodic motion from just a small number of components. Thus, we need to save only a few sets of frequency, amplitude and phase, instead of the time sequence, which solves the memory problem.

We represented the responses by four Fourier components and made an animation of the bamboo swaying. The synthesized animation is almost visually identical to the original one. Even when the two sequences are presented side by side on the same monitor simultaneously, there is little possibility of recognizing any difference. Figures 6(a)–6(c) show the motion reconstructed from the four frequency components. It has been demonstrated that the four components are good enough for the motion of the trunk and the branch to be reconstructed accurately. The reconstruction for the leaf produced relatively large errors (6 per cent), since the leaf response has a broader bandwidth. This means that we need more components (say 16) to increase reconstruction accuracy. Fortunately, however, because leaf motion is fast and very small in amplitude, these errors are hard to perceive from the animation.

4. CONCLUSIONS

We have proposed a simple, stable and efficient method to compute periodic motion from linear equation systems. The method first calculates the response of the system when the external periodic force is applied during one period. It then sums up the periodically shifted versions of the response to yield the periodic solution. We have also shown that Fourier decomposition is very effective in compressing motion data without loss of significant fidelity.

ACKNOWLEDGEMENTS

We would like to thank Yukio Tokunaga for his administrative support and encouragement, Atsushi Kajiyama for his technical support, and Toki Takahashi and Takafumi Saito for helpful discussion.

REFERENCES

1. A. Witkin, D. Baraff and M. Kass, 'An introduction to physically based modeling', *SIGGRAPH '95 Course Note 34*, 1995.
2. B. Reeves and R. Blau, 'Approximate and probabilistic algorithms for shading and rendering structured particle systems', *Computer Graphics (SIGGRAPH '85)*, **19**(3), 312–322 (1985).
3. J. Wejchert and D. Haumann, 'Animation aerodynamics', *Computer Graphics (SIGGRAPH '91)*, **25**(4), 19–22 (1991).
4. M. Shinya and A. Fournier, 'Stochastic motion—motion under the influence of wind', *Proceedings of Eurographics '92*, 1992, pp. C-119–C-128.
5. J. Stan and E. Fiume, 'Turbulent wind fields for gaseous phenomena', *Computer Graphics (SIGGRAPH '93), Ann. Conf. Ser.*, 369–376 (1993).
6. L. Ling, M. Damodaran and R. Gay, 'Aerodynamic force models for animating cloth motion in air flow', *The Visual Computer*, **12**, 84–104 (1996).
7. E. Simiu and R. H. Scanlan, *Wind Effects on Structures: An Introduction to Wind Engineering*, Wiley, New York, 1978.