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COMPUTER AIDED GEOMETRIC DESIGN

Surface deformation with differential geometric structures

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Abstract

This paper considers the deformation of a given surface to a surface that smoothly connects to previously designed surfaces while reflecting the overall shape of the initial surface. We introduce deformation energy using a Laplacian-based functional, which is defined by the global differential geometric structures of the initial surface. It is shown that the proposed deformation energy does not depend on representations of the initial surface, and relates to the mean curvature vector, a geometric quantity correlated to overall surface shape, and also has a good computational property. An example is presented to demonstrate the effectiveness of our method.

Keywords: Surface deformation; Smooth connection; Deformation energy; Laplacian; Differential geometric structure; Mean curvature vector; Constrained optimization

1. Introduction

Modeling complex surface shapes (e.g., human bodies) can be simplified by assembling several previously designed surfaces (e.g., head, arm, leg, etc.) with appropriate deformation. For functional or aesthetic reasons, surface modeling often requires smooth connections of adjoining surfaces. This paper considers deformation of a given surface M such that the deformed surface M' smoothly connects to designed surfaces $M_1, \dots, M_L$ positioned in three dimensional Euclidean space $\mathbb{R}^3$.

Since the imposed design condition is only a boundary condition, in general, the possible solutions have an infinite degree of freedom. However, practical methods usually present a deformation family $\mathcal{F}$ of M with a finite degree of freedom if representations of

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M and $M_1, \ldots, M_L$ are given (see, for example, (Peters, 1990)). For instance, in tensor product spline representations such as Bézier or B-spline, control points, except those around the boundary, are used as the degree of freedom to manipulate such deformation of M.

Interactive manipulation for such deformation of M becomes difficult with an increase in the remaining degree of freedom that is not constrained by the imposed design condition, (for example, with an increase in complexity of M in tensor product spline representation). Hence, it is desired to automatically control this extra degree of freedom for such deformation. In this paper, we discuss a method of choosing one deformation map in the family $\mathcal{F}$ of admissible deformation maps of M such that the shape of the deformed surface M' is as close to the shape of the initial surface M as possible under a reasonable measure.

Such a problem is usually formulated by deformation energy and solved as a minimization problem. In our problem, the following claims are addressed for deformation energy:

- *Geometric property*: Deformation energy should be defined by the geometric structures of M and should not depend on the representation of M. Moreover, it should relate to one geometric quantity reflecting overall surface shape.
- *Computational property*: There should uniquely exist a deformation map of M that minimizes deformation energy in the deformation family $\mathcal{F}$, and the minimizing deformation map should be explicitly calculated.

For the deformation energy satisfying both these geometric and computational properties, we propose a Laplacian-based energy functional. The proposed energy functional is defined by the global differential geometric structures of the initial surface M. It is shown that the proposed deformation energy relates to the mean curvature vector, a geometric quantity correlated to overall surface shape, and also satisfies our computational property.

Section 2 briefly summarizes related works. In Section 3, the energy functional is proposed and its geometric meaning is discussed. Section 4 describes its computation nature, that is, its energy minimization. In Section 5, when the initial surface M is a uniform bicubic B-spline surface, a computation to obtain optimal deformation is provided and an experimental result is presented.

2. Related work

Variational methods have recently become popular in surface modeling. There are two types of applications: surface creation and surface deformation. In surface creation, users only specify constraints such as boundary conditions and the methods determine the optimal surfaces that satisfy the constraints. In surface deformation, on the other hand, users specify an initial surface as well as boundary conditions, and the methods try to find a surface that reflects the overall shape of the initial surface while fulfilling given constraints.

In surface creation, much attention has been paid to the selection of an appropriate fairness functional. Based on a physical analogy or geometric property, several fairness functionals have been proposed. The functional of Moreton and Séquin minimizes the variation of curvature and produces high quality surfaces with predictable behavior (see (Moreton and Séquin, 1992)). This works fine, but its computational cost is quite high. To reduce the computational cost, Greiner proposed a Laplacian-based functional (see (Greiner, 1994)), which is similar to our deformation energy (see (Kimura and Saito, 1991)). He also showed that in the context of boundary value problems, his fairness functional supplies a kind of approximation of an equilibrium surface under the enegy of a thin plate, and is also based on geometric consideration.

Our attention is devoted to surface deformation. Namely, we consider energy functionals to generate deformed surfaces that reflect the overall shapes of initial surfaces. The following energy functionals have been proposed:

- The L^2-norm of the displacement amount of the surface (see (Lott and Pullin, 1988)).
- The sum of weighted L^2-norms of the differences of first fundamental forms and second fundamental forms (see (Terzopoulos et al., 1987)).

Although the former deformation energy does not depend on surface representation and satisfies our computational property, it does not relate to a geometric quantity reflecting surface shape. On the other hand, although the latter deformation energy has a strong geometric interpretation [1] and also has a physical interpretation [2], it yields a difficult nonlinear optimization problem. In practice, this energy functional should be approximated from the physical point of view (see, for example, (Terzopoulos et al., 1987)) or simply replaced by an energy functional analogous to a thin-plate-under-tension model (see (Welch and Witkin, 1992)). In these cases, however, the energy fuctionals depend on surface representation, and thus, they are not associated with the geometric structures of the initial surface.

3. Deformation energy

This and the following sections are described in terms of the manifold theory (see, for example, (Matsushima, 1972; Hirsh, 1982)), and we will use the same notations throughout this paper. Since the surfaces to be considered are smooth and have piecewise C^∞-parametric representations, we assume that they are C^∞-submanifolds of the C^∞-manifold $\mathbb{R}^3$ in the theory to be developed. In other words, for simplicity, the theory is developed in the C^∞-category although it is sufficient in the C^2-category.

3.1. Laplacian

First, a definition of the Laplacian is briefly reviewed (see, for example, (Chavel, 1984)). It is a natural generalization of the usual Laplacian $\partial^2/\partial(u^1)^2 + \partial^2/\partial(u^2)^2$ on Euclidean space.

[1] Surfaces that have the same first and second fundamental forms have the same shape.
[2] Deformation of a surface simulates deformation of an elastic membrane.

Let M be a compact submanifold of $\mathbb{R}^3$ and ι the inclusion map of M to $\mathbb{R}^3$. The canonical metric $\langle\, , \rangle$ on $\mathbb{R}^3$ induces a Riemannian metric on M. The Laplacian Δ of the Riemannian manifold $(M, \langle\, , \rangle)$ is the following elliptic differential operator on M:

$$\Delta f = \operatorname{div} \operatorname{grad} f,$$

where f is a C^∞-function on M. grad f is a vector field on M defined by

$$\langle (\operatorname{grad} f)_p, v\rangle = vf$$

for all $v \in T_pM$ (the tangent space of M at $p \in M$), where vf is the derivative of f at p in the v-direction. Note that the largest increase in f occurs in the grad f-direction and the degree of growth of f in this direction is the magnitude of grad f. For a C^∞-vector field X, div X is the following function on M:

$$L_X\, dA = (\operatorname{div} X)\, dA,$$

where dA is the area element of the Riemannian manifold $(M, \langle\, , \rangle)$ and $L_X\, dA$ is the Lie derivative of dA with respect to X. Note that div X measures the infinitesimal distortion of area by the flow $\{\xi_t\}$ on M generated by X, that is, for any closed subset K of M,

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\xi_t(K)} dA = \int_K \operatorname{div} X\, dA.$$

For any point p in M, there exists a neighborhood U of p in M, an open set W in $\mathbb{R}^2$ and a non-singular one-to-one C^∞-map x from W onto $U \subset \mathbb{R}^3$. Accordingly, Δf is locally expressed on U as follows:

$$(\Delta f)\circ x = \sum_{1\leqslant\lambda,\mu\leqslant 2} g^{\lambda\mu}\frac{\partial^2(f\circ x)}{\partial u^\lambda \partial u^\mu} + \sum_{1\leqslant\lambda,\mu\leqslant 2}\left(\frac{\partial g^{\lambda\mu}}{\partial u^\lambda} + \frac{1}{2}g^{\lambda\mu}\frac{\partial \log g}{\partial u^\lambda}\right)\frac{\partial(f\circ x)}{\partial u^\mu}, \tag{3.1}$$

where (u^1, u^2) is the canonical coordinate system of $\mathbb{R}^2$, $\{g_{\lambda\mu}\}$ are the components of the Riemannian metric tensor (the first fundamental form) $\langle\, , \rangle$ on M with respect to the chart (U, x), that is, $g_{\lambda\mu} = \langle \partial x/\partial u^\lambda, \partial x/\partial u^\mu\rangle$, $\{g^{\lambda\mu}\}$ are the components of the inverse matrix of the matrix $(g_{\lambda\mu})$, and g is the determinant of the matrix $(g_{\lambda\mu})$.

Moreover, the Laplacian satisfies the following property (see, for example, (Matsushima, 1972)).

Theorem 3.1 (The maximum principle). *Suppose f is a C^∞-function on M such that $\Delta f \equiv 0$ on M. Then, if f is not a constant, f does not achieve its maximum and minimum in the interior of M.*

We remark that the Laplacian Δ is a differential geomeric structure of the submanifold M of $\mathbb{R}^3$ and does not depend on the representation of M.

3.2. Proposal of energy functional

The usual norm $|\cdot|$ on $\mathbb{R}^3$ defines a pointwise norm on M. The vector space $C^\infty(M; T\mathbb{R}^3)$ consisting of all $\mathbb{R}^3$-valued C^∞-vector fields on M has the following inner product and associated norm:

$$(X,Y) = \int_M \langle X,Y\rangle \, dA, \qquad \|X\|^2 = (X,X) = \int_M |X|^2 \, dA,$$

for $X, Y \in C^\infty(M; T\mathbb{R}^3)$.

We consider the submanifold M as the surface to be deformed. Let ϕ be a deformation map of the submanifold M, that is, a C^∞-map from M to $\mathbb{R}^3$. Our proposed deformation energy $\mathcal{E}(\phi)$ for the deformation map ϕ is defined as follows:

$$\mathcal{E}(\phi) = \|\Delta\phi - \Delta\iota\|^2.$$

Hence, the proposed deformation energy is defined by the global differential geometric structures of M and does not depend on the representation of M.

3.3. Geometric meaning

Let H be the mean curvature vector of M (see, for example, (Palais and Terng, 1988; Gallot et al., 1990)). Note that H is a normal vector field on M, and the deformation of M generated from H yields the largest decrease in area (see Appendix). For example, if M is a sphere of radius r centered at the origin in $\mathbb{R}^3$, $H_p = -2p$ for any $p \in M$. Let ν be the inward unit normal vector field on M, that is, $\nu = H/|H|$ at each point such that $H \neq 0$. The pointwise magnitude $|H|$ of the vector field H is twice the mean curvature $(1/2)\langle H, \nu\rangle$ of M with respect to ν. At each point of M, the mean curvature of M measures, on average, how M curves to the ν-direction. It is concluded that the mean curvature vector of a surface is one of the geometric quantities correlated to the overall shape.

We assume that $M' = \phi(M)$ is a submanifold of $\mathbb{R}^3$. Note that the metric $\langle\, ,\rangle$ also defines a Riemannian metric on M'. Let H' be the mean curvature vector of the Riemannian manifold $(M', \langle\, ,\rangle)$ and Δ' the Laplacian of the Riemannian manifold $(M, \phi^*\langle\, ,\rangle)$, where $\phi^*\langle\, ,\rangle$ is the Riemannian metric on M induced from the canonical metric $\langle\, ,\rangle$ on $\mathbb{R}^3$ by the map ϕ.

Theorem 3.2. *Suppose the deformation map ϕ of M is an isometric embedding, that is, ϕ is a one-to-one C^∞-map such that $\phi^*\langle\, ,\rangle = \langle\, ,\rangle$ on M. Then,*

$$\mathcal{E}(\phi) = \|(H' \circ \phi) - H\|^2.$$

Proof. It is known that $H = \Delta\iota$ and $H' \circ \phi = \Delta'\phi$ (see, for example, (Palais and Terng, 1988; Gallot et al., 1990)). For a chart (U, x) of M, the components of the Riemannian metric tensor $\phi^*\langle\, ,\rangle$ on M are denoted by $\{g'_{\lambda\mu}\}$. For a C^∞-function f on M, the Laplacian $\Delta' f$ is locally expressed on U as follows (see Eq. (3.1)):

$$(\Delta' f)\circ x = \sum_{\lambda,\mu} g'^{\lambda\mu}\frac{\partial^2(f\circ x)}{\partial u^\lambda \partial u^\mu} + \sum_{\lambda,\mu}\left(\frac{\partial g'^{\lambda\mu}}{\partial u^\lambda} + \frac{1}{2}g'^{\lambda\mu}\frac{\partial \log g'}{\partial u^\lambda}\right)\frac{\partial(f\circ x)}{\partial u^\mu},$$

where $(g'^{\lambda\mu}) = (g'_{\lambda\mu})^{-1}$ and $g' = \det(g'_{\lambda\mu})$. Since $\phi^*\langle\, ,\,\rangle = \langle\, ,\,\rangle$, $g'_{\lambda\mu} = g_{\lambda\mu}$. Hence, $\Delta'\phi = \Delta\phi$. This result proves the theorem. □

If only those deformation maps of M that do not involve stretching and shrinking are considered, the proposed deformation energy measures the total difference between the mean curvature vectors of M and a deformed surface (see Theorem 3.2). The proposed deformation energy approximately measures the global geometric quantity as above for small stretching and shrinking deformation maps (see the proof of Theorem 3.2). Our energy functional is precisely defined by the global geometric structures of M. Thus, the deformation energy relates to one geometric quantity correlated to overall surface shape.

Hence, the proposed deformation energy satisfies our geometric property.

4. Energy minimization

Energy minimization of the proposed deformation energy is discussed within a family $\mathcal{F}$ having a finite degree of freedom, where $\mathcal{F}$ consists of deformation maps of M to surfaces that smoothly connect to designed surfaces $M_1, \ldots, M_L$ positioned in $\mathbb{R}^3$.

Suppose a special solution $\phi_0 \in \mathcal{F}$ is given. Then, $\mathcal{F} = \phi_0 + \mathcal{F}_0$, where $\mathcal{F}_0$ is a finite dimensional subspace of the vector space $\{\phi \in C^\infty(M, \mathbb{R}^3) \mid \phi \equiv 0 \text{ on } \partial M\}$, where $C^\infty(M, \mathbb{R}^3)$ is the vector space consisting of C^∞-maps from M to $\mathbb{R}^3$. Let $\{\psi_1, \ldots, \psi_N\}$ be a basis of the vector space $\mathcal{F}_0$. We get the following theorem:

Theorem 4.1. *In the deformation family $\mathcal{F}$, the energy functional $\mathcal{E}$ has a unique minimum $\tilde{\phi}$, which is explicitly expressed as follows:*

$$\tilde{\phi} = \phi_0 + \sum_{s=1}^{N} c^s \psi_s, \tag{4.1}$$

where

$$\begin{pmatrix} c^1 \\ \vdots \\ c^N \end{pmatrix} = ((\Delta\psi_r, \Delta\psi_s))^{-1} \begin{pmatrix} -(\Delta(\phi_0 - \iota), \Delta\psi_1) \\ \vdots \\ -(\Delta(\phi_0 - \iota), \Delta\psi_N) \end{pmatrix}. \tag{4.2}$$

Proof. By $\mathcal{F} \ni \phi_0 + \sum_{s=1}^{N} y^s\psi_s \leftrightarrow (y^1, \ldots, y^N) \in \mathbb{R}^N$, $\mathcal{F}$ is identified with $\mathbb{R}^N$. From this correspondence, $\mathcal{E}$ can be regarded as a function on $\mathbb{R}^N$. For $1 \leqslant r, s \leqslant N$,

$$\frac{\partial \mathcal{E}}{\partial y^s}(y^1, \ldots, y^N) = 2\sum_{r=1}^{N}(\Delta\psi_r, \Delta\psi_s)y^r + 2(\Delta(\phi_0 - \iota), \Delta\psi_s), \tag{4.3}$$

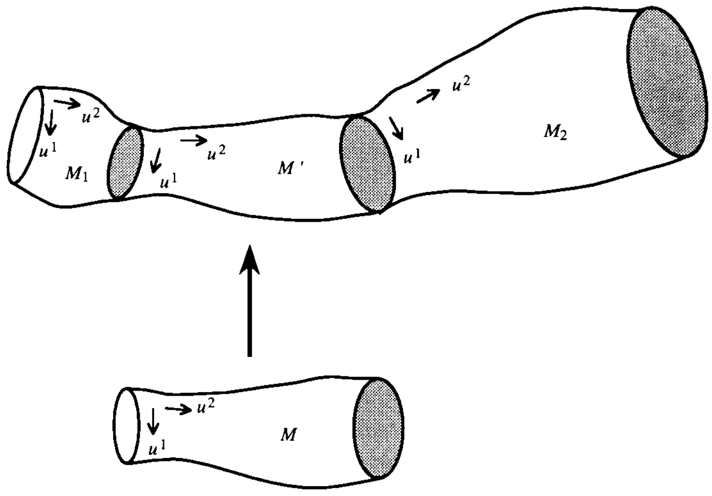

Fig. 1. The configuration of deformed B-spline surface M' of M and the B-spline surfaces M_1, M_2.

$$\frac{\partial^2 \mathcal{E}}{\partial y^r \partial y^s}(y^1, \ldots, y^N) = 2(\Delta\psi_r, \Delta\psi_s). \tag{4.4}$$

Assume $\sum_{s=1}^{N} d^s \Delta\psi_s \equiv 0$ on M for $d^1, \ldots, d^N \in \mathbb{R}$. From the definition of $\mathcal{F}_0$, $\sum_{s=1}^{N} d^s \psi_s \equiv 0$ on ∂M. By virtue of the maximum principle (see Theorem 3.1), these yield $\sum_{s=1}^{N} d^s \psi_s \equiv 0$ on M. Hence $d^1 = \cdots = d^N = 0$. This shows that $\Delta\psi_1, \ldots, \Delta\psi_N$ are linearly independent in the vector space $C^\infty(M, \mathbb{R}^3)$. Thus the matrix $((\Delta\psi_r, \Delta\psi_s))$ is a positive definite symmetric matrix. Hence, from Eq. (4.3), the following system of equations has a unique solution $(c^1, \ldots, c^N)$ given by Eq. (4.2): $\partial\mathcal{E}/\partial y^s \equiv 0$ for $1 \leqslant s \leqslant N$. By Eq. (4.4), it turns out that the Hessian of $\mathcal{E}$ is always positive definite on $\mathbb{R}^N$. These results prove our theorem. □

This theorem shows that the proposed deformation energy satisfies our computational property.

5. Application to B-spline surfaces

B-spline representation is widely used in surface modeling because of its good features in local shape control and smooth surface composition (see, for example, (Farin, 1990; Foley et al., 1990; Hoschek and Lasser, 1993)). In this section, the energy minimization procedure is explicitly given for an initial surface M and adjacent surfaces M_1, M_2 all represented as non-singular cylinder-like uniform bicubic B-spline surfaces (see Fig. 1). We assume that the B-spline surfaces M, M_1 and M_2 are respectively defined by the control meshes $\{P_{ij};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n+3\}$, $\{P_{ij}^{(1)};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n_1+3\}$ and $\{P_{ij}^{(2)};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n_2+3\}$ such that

$$P_{i+m+1\,j} = P_{ij}, \quad (0 \leqslant j \leqslant n+3),$$
$$P^{(1)}_{i+m+1\,j} = P^{(1)}_{ij} \quad (0 \leqslant j \leqslant n_1+3),$$
$$P^{(2)}_{i+m+1\,j} = P^{(2)}_{ij} \quad (0 \leqslant j \leqslant n_2+3),$$

for $i = 0, 1, 2$, where $3 \leqslant m, n \in \mathbb{Z}$, $0 \leqslant n_1, n_2 \in \mathbb{Z}$.

Note that $M = x([0, m+1] \times [0, n+1])$, where

$$x(u^1, u^2) = \sum_{i=0}^{m+3} \sum_{j=0}^{n+3} P_{ij} N_{i,4}(u^1) N_{j,4}(u^2), \tag{5.1}$$

and $N_{k,4}$ is a B-spline function of order 4 with respect to the knot sequence $\mathbb{Z}$ such that $\operatorname{supp} N_{k,4} = [k-3, k+1]$ (see, for example, (Farin, 1990; Foley et al., 1990; Hoschek and Lasser, 1993)). Note also that the map $x : [0, m+1) \times [0, n+1) \to \mathbb{R}^3$ is one-to-one and C^2, and the Jacobian matrix has full rank.

The configuration of the surfaces M_1, M_2 and deformed surface M' of M is illustrated in Fig. 1, where M' and M_1, M_2 smoothly connect along their boundary curves. In this case, the deformation family $\mathcal{F}$ of M to be considered is constructed from the deformation maps controlled by the B-spline control points of M.

5.1. Deformation family

We construct the family $\mathcal{F} = \phi_0 + \mathcal{F}_0$ of the admissible deformation maps controlled by the B-spline control points of M. The control points of M corresponding to the mesh

$$\Omega_0 = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant m+3,\ (0 \leqslant j \leqslant 2 \text{ or } n+1 \leqslant j \leqslant n+3)\}$$

are used to achieve a smooth connection of M' to M_1, M_2, and the control points of M corresponding to the mesh

$$\Omega = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leqslant i \leqslant m,\ 3 \leqslant j \leqslant n\}$$

are free parameters to control the admissible deformation maps of M.

When combining two cylinder-like spline surfaces, correspondence of control points between the surfaces must be specified. We assume that such correspondence is given for the surface M and the surfaces M_1, M_2, and conforming with this correspondence, the deformation family $\mathcal{F}$ is constructed as follows: An element ϕ of $\mathcal{F}$ generates a cylinder-like uniform bicubic B-spline surface $M' = \phi(M)$ defined by the control mesh $\{P'_{ij};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n+3\}$ such that $P'_{i+m+1\,j} = P'_{ij}$, $(0 \leqslant i \leqslant 2,\ 0 \leqslant j \leqslant n+3)$,

$$P'_{ij} = \begin{cases} P^{(1)}_{i\,j+n_1+1} & (0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant 2) \\ P^{(2)}_{i\,j-n-1} & (0 \leqslant i \leqslant m+3,\ n+1 \leqslant j \leqslant n+3), \end{cases} \tag{5.2}$$

(see Fig. 2).

A special solution ϕ_0 and a basis of the vector space $\mathcal{F}_0$ are explicitly expressed as functions on M to exploit the result of the previous section. We define a C^2-function f_{ij} $(0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n+3)$ on M as follows:

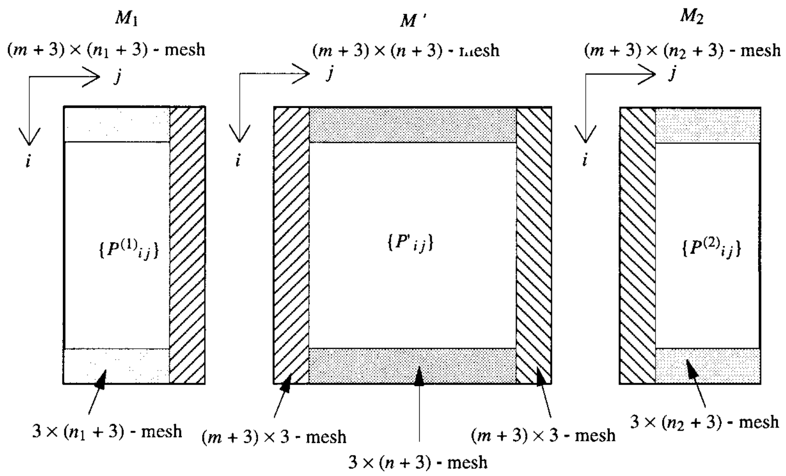

Fig. 2. The B-spline control meshes of M_1, M' and M_2: The meshes of the same pattern show the same array of the B-spline control points.

$$f_{ij}(x(u^1,u^2)) = \begin{cases} (N_{i,4}(u^1) + N_{i+m+1,4}(u^1))N_{j,4}(u^2) & (0 \leqslant i \leqslant 2) \\ N_{i,4}(u^1)N_{j,4}(u^2) & (3 \leqslant i \leqslant m), \end{cases} \tag{5.3}$$

for $(u^1,u^2) \in [0,m+1] \times [0,n+1]$ (see Eq. (5.1), Fig. 3). These are the lifts of the B-spline basis functions to M, and the inclusion map ι of M to $\mathbb{R}^3$ can be written as

$$\iota = \sum_{0\leqslant i\leqslant m,\, 0\leqslant j\leqslant n+3} P_{ij} f_{ij}. \tag{5.4}$$

Note that the support of f_{ij} is small. This makes the matrix $((\Delta\psi_r, \Delta\psi_s))$ in Eq. (4.2) very sparse, and the computational quantities of $(\Delta\psi_r, \Delta\psi_s)$'s and $(\Delta(\phi_0 - \iota), \Delta\psi_s)$'s in Eq. (4.2) decrease (see Eqs. (5.9)–(5.11)).

Let $\{Q_{ij};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n+3\}$ be the control mesh of the cylinder-like uniform bicubic B-spline surface $\phi_0(M)$. The special solution ϕ_0 and the vector space $\mathcal{F}_0$ are then given as follows:

$$\phi_0 = \sum_{0\leqslant i\leqslant m,\, 0\leqslant j\leqslant n+3} Q_{ij} f_{ij}, \tag{5.5}$$

$$\mathcal{F}_0 = \left\{ \phi \in C^2(M,\mathbb{R}^3) \;\middle|\; \phi = \sum_{(i,j)\in\Omega} V_{ij} f_{ij},\ V_{ij} \in \mathbb{R}^3 \right\}.$$

For $(i,j) \in \Omega$ and $a = 1,2,3$, $\phi^a_{ij} \in C^2(M,\mathbb{R}^3)$ is defined by

$$\phi^1_{ij} = (f_{ij},0,0), \qquad \phi^2_{ij} = (0,f_{ij},0), \qquad \phi^3_{ij} = (0,0,f_{ij}). \tag{5.6}$$

Let $\{\psi_1,\ldots,\psi_{3(m+1)(n-2)}\}$ be the sequence generated by $\{\phi^a_{ij}\}_{(i,j)\in\Omega, 1\leqslant a\leqslant 3}$, that is,

$$\psi_{s(i,j,a)} = \phi^a_{ij}, \tag{5.7}$$

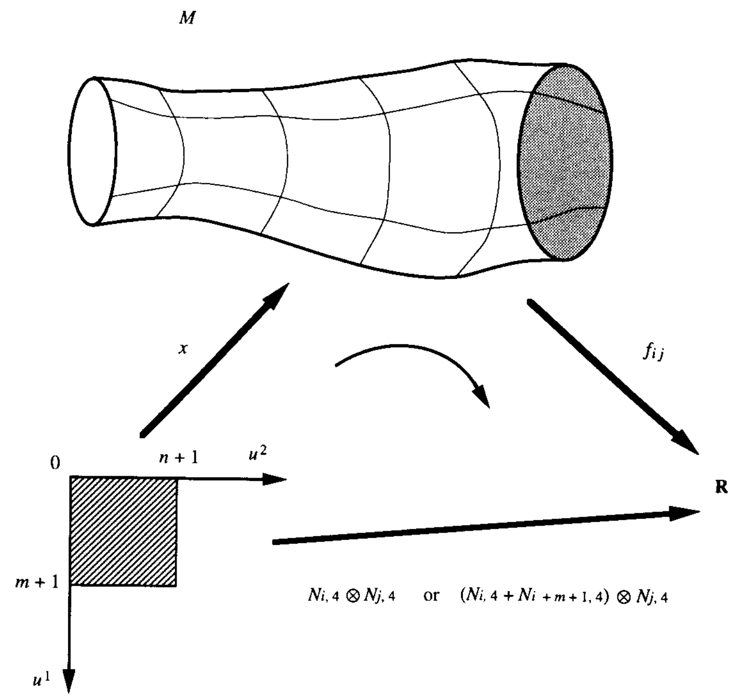

Fig. 3. The lift f_{ij} of the B-spline basis function to M.

where $s(i,j,a) = (a-1)(m+1)(n-2)+i(n-2)+j-2$. Then, $\{\psi_1, \dots, \psi_{3(m+1)(n-2)}\}$ is a basis of $\mathcal{F}_0$.

5.2. Calculation of optimal deformation

From Theorem 4.1 and Eqs. (5.5)–(5.7), the energy minimizing deformation map $\tilde{\phi}$ is obtained as follows:

$$\tilde{\phi} = \sum_{(i,j)\in\Omega_0} Q_{ij} f_{ij} + \sum_{(i,j)\in\Omega} (Q_{ij} + V_{ij}) f_{ij}, \tag{5.8}$$

where

$$V_{ij} = (c^{s(i,j,1)}, c^{s(i,j,2)}, c^{s(i,j,3)}), \quad (i,j) \in \Omega,$$

and $(c^1, \dots, c^{3(m+1)(n-2)})$ is derived from Eq. (4.2). Note that the B-spline surface $\tilde{\phi}(M)$ is defined by the control mesh $\{\widetilde{P}_{ij};\ 0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n+3\}$ such that $\widetilde{P}_{i+m+1\,j} = \widetilde{P}_{ij}$, $(0 \leqslant i \leqslant 2,\ 0 \leqslant j \leqslant n+3)$,

$$\widetilde{P}_{ij} = \begin{cases} Q_{ij}, & (i,j) \in \Omega_0 \\ Q_{ij} + V_{ij}, & (i,j) \in \Omega. \end{cases}$$

To obtain the deformation map $\tilde{\phi}$ of M, the square matrix $((\Delta\psi_r, \Delta\psi_s))$ of degree $3(m+1)(n-2)$ and $(\Delta(\phi_0 - \iota), \Delta\psi_s)$'s have to be calculated (see Theorem 4.1).

First, calculation of the matrix $\big((\Delta\psi_r, \Delta\psi_s)\big)$ is described. From Eq. (5.6) and Eq. (5.7), it is easily seen that

$$((\Delta\psi_r, \Delta\psi_s)) = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B} \end{pmatrix},$$

where $\boldsymbol{0}$ is the zero square matrix of degree $(m+1)(n-2)$ and $\boldsymbol{B} = (b_{rs})$ is the square matrix of degree $(m+1)(n-2)$ such that

$$b_{s(i,j,1)s(k,l,1)} = \int_M \Delta f_{ij} \Delta f_{kl}\, dA \tag{5.9}$$

for (i,j), $(k,l) \in \Omega$.

Next, calculation of the vector $((\Delta(\phi_0 - \iota), \Delta\psi_s))$ is described. $(\Delta(\phi_0 - \iota), \Delta\phi^a_{ij})$ has to be calculated for $(i,j) \in \Omega$ and $1 \leqslant a \leqslant 3$ (see Eq. (5.7)). From Eqs. (5.4)–(5.6), the following is obtained:

$$(\Delta(\phi_0 - \iota), \Delta\phi^a_{ij}) = \sum_{0 \leqslant k \leqslant m,\, 0 \leqslant l \leqslant n+3} R^a_{kl} \int_M \Delta f_{ij} \Delta f_{kl}\, dA, \tag{5.10}$$

where $R_{kl} = Q_{kl} - P_{kl}$ and $R_{kl} = (R^1_{kl}, R^2_{kl}, R^3_{kl})$.

Hence, it is required that the integral $\int_M \Delta f_{ij} \Delta f_{kl}\, dA$ be computed for $(i,j) \in \Omega$, $0 \leqslant k \leqslant m$, $0 \leqslant l \leqslant n+3$ (see Eqs. (5.9), (5.10)). The integral is zero for $3 < |i-k| < m-2$ or $3 < |j-l|$ since the supports of f_{ij} and f_{kl} do not intersect, and for other i, j, k, l, it can be computed by

$$\int_0^{m+1} du^1 \int_0^{n+1} du^2\, (\Delta f_{ij} \circ x)(u^1, u^2)\, (\Delta f_{kl} \circ x)(u^1, u^2)\, \sqrt{g(u^1, u^2)}, \tag{5.11}$$

where $g_{\lambda\mu} = \langle \partial x/\partial u^\lambda, \partial x/\partial u^\mu \rangle$ can be calculated from Eq. (5.1), $g = \det(g_{\lambda\mu})$, $(g^{\lambda\mu}) = (g_{\lambda\mu})^{-1}$, and then $\Delta f_{ij} \circ x$ can be calculated from Eq. (3.1). Note that the integration domain becomes small since suppf_{ij} and suppf_{kl} are small. Note also that the numerical values of the integrand can be efficiently computed by using matrix operations since a uniform bicubic B-spline surface is locally represented by a bicubic polynomial.

Hence, it is feasible to compute the optimal deformation map $\tilde{\phi}$ of M for the cylinder-like uniform bicubic B-spline surfaces M_1, M_2 and M (see Eq. (5.8)). Application to other B-spline surfaces of different topologies is possible in a similar way.

5.3. Example

An example using our method is presented in the case of the cylider-like uniform bicubic B-spline surfaces described above.

Fig. 4(a) shows the designed surfaces M_1, M_2 (the green surfaces) positioned in $\mathbb{R}^3$ and the initial surface M (the white surface) to be deformed and set between M_1 and M_2, where $m = 3$, $n = 5$, $n_1 = 3$ and $n_2 = 4$. In Fig. 4(b), a special solution $\phi_0(M)$ is shown by the white surface (see Eq. (5.5)), where each control point Q_{ij}, $(0 \leqslant i \leqslant m+3,\ 0 \leqslant j \leqslant n+3)$ is obtained by Eq. (5.2) for (i,j) with $j = 0, 1, 2, n+1, n+2, n+3$, and is the same as the contol point of M for other (i,j). Fig. 4(c) shows the result of applying our method, where the white surface represents the deformed surface of M under the optimal deformation map $\tilde{\phi}$.

This result visually demonstrates that the optimal deformed surface $\tilde{\phi}(M)$ is much improved from the special solution $\phi_0(M)$, and the deformed surface $\tilde{\phi}(M)$ reflects the overall shape of the initial surface M while fulfilling the smooth connection constraint. In particular, it is seen that the optimal deformed surface inherits a swelling from the initial surface.

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Appendix

Let M be a compact submanifold of $\mathbb{R}^3$ and $\Phi : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^3$ a variation of M with fixed boundary, that is, Φ is a C^∞-map such that

$$\Phi(0,p) = p \quad \text{for } p \in M,$$
$$\Phi(t,p) = p \quad \text{for } -\varepsilon < t < \varepsilon, \quad p \in \partial M.$$

Put $\phi_t := \Phi(t,\cdot)$ for $-\varepsilon < t < \varepsilon$. Then ϕ_t is a local diffeomorphism from M to $\phi_t(M)$ for sufficiently small t. The following equation holds (see, for example, (Gallot et al., 1990)):

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(\phi_t(M)) = -\int_M \langle H, Y\rangle\, dA,$$

where H is the mean curvature vector of M, dA is the area element of M and Y is the variation vector field associated to Φ, that is,

$$Y_p = \frac{d}{dt}\bigg|_{t=0} \phi_t(p)$$

for $p \in M$. Hence, if a variation Φ of M has the variation vector field that coincides with H, the induced flow $\{\phi_t\}$ causes the largest decrease in area of M.

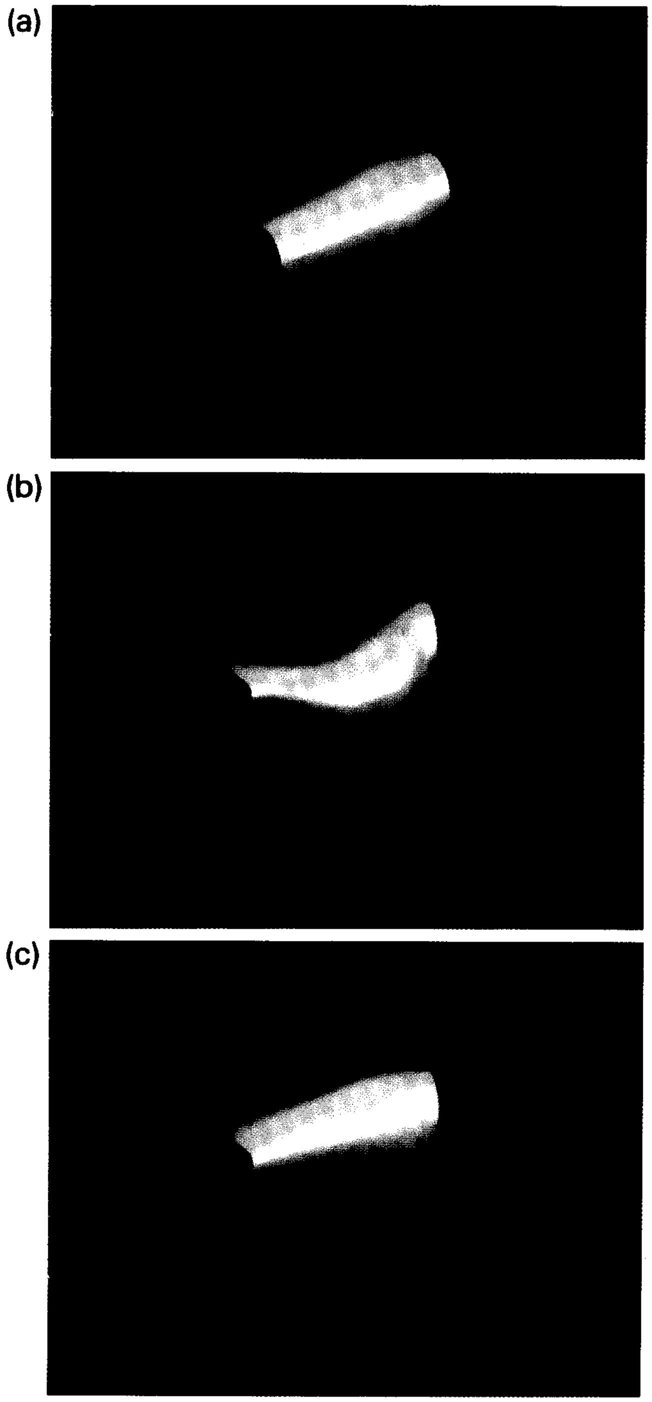

Fig. 4. (a) The initial surface. (b) A special solution. (c) The optimal deformed surface.

References

Chavel, I. (1984), *Eigenvalues in Riemannian Geometry*, Academic Press, New York.

Farin, G. (1990), *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, Boston, MA.

Foley, J., van Dam, A., Feiner, S. and Hughes, J. (1990), *Computer Graphics*, Addison-Wesley, New York.

Gallot, S., Hulin, D. and Lafontaine, J. (1990), *Riemannian Geometry*, Universitext, Springer, Berlin.

Greiner, G. (1994), Surface construction based on variational principles, in: P. Laurent, A. Le Méhauté and L. Schumaker, eds., *Wavelets, Images and Surface Fitting*, A K Peters, Wellesley, MA.

Hirsh, M. (1982), *Differential Topology*, Graduate Texts in Mathematics, Springer, New York.

Hoschek, J. and Lasser, D. (1993), *Fundamentals of Compuer Aided Geometric Design*, A K Peters, Wellesley, MA.

Kimura, M. and Saito, T. (1991), A generating method of blending surfaces using physical phenomena, in: *Proc. 42nd National Convention IPS Japan* (2), 339–340, in Japanese.

Lott, N. and Pullin, D. (1988), Method for fairing B-spline surfaces, Computer Aided Design 20, 597–604.

Matsushima, Y. (1972), *Differentiable Manifolds*, Marcel Dekker, New York.

Moreton, H. and Séquin, C. (1992), Functional optimization for fair surface design, Computer Graphics 26 (2) (Proc. SIGGRAPH'92), 167–176.

Palais, S. and Terng, C. (1988), *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Mathematics, Springer, Berlin.

Peters, J. (1990), Local smooth surface interpolation: a classification, Computer Aided Geometric Design 7, 191–195.

Terzopoulos, D., Platt, J., Barr, A. and Fleischer. K. (1987), Elastically deformable models, Computer Graphics 21 (4) (Proc. SIGGRAPH'87), 205–214.

Welch, W. and Witkin, A. (1992), Variational surface modeling, Computer Graphics 26 (2) (Proc. SIGGRAPH'92), 157–166.